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UNIQUE GLOBAL SOLUTION WITH RESPECT TO TIME OF INITIAL-BOUNDARY VALUE PROBLEMS FOR ONE-DIMENSIONAL EQUATIONS OF A VISCOUS GAS

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A system of equations of a viscous heat-conducting perfect gas [1, 2] is studied for the case of a one-dimensional motion with plane waves. Unique solvability of the problem of gas flow in a bounded region with impermeable thermally insulated boundaries is proved in the classes of the generalized (strong) and classical solutions. The theorem of existence is established by the method of continuing the local solution with respect to time, based on the a priori global estimates. A major role is played here by the upper and lower bounds for the density and temperature. A method of obtaining these bounds was worked out in [3] for a simpler problem of gas expanding into vacuum. The problem of existence of global solutions of one-dimensional nonstationary equations of a viscous compressible gas was dealt with in [4—6] only for the simplest models.

1. Formulation of the problem. The system of equations of a viscous gas can be written in the Lagrange variables in the following form (see [2], ch. 2):

$$\frac{\partial V}{\partial \tau}-\frac{\partial u}{\partial q}=0,\quad \frac{\partial u}{\partial \tau}=\frac{\partial}{\partial q}\left(\mu V^{-1}\frac{\partial u}{\partial q}-p\right) \tag{1.1}$$

$$\frac{\partial}{\partial \tau}\left(\varepsilon+\frac{u^2}{2}\right)=\frac{\partial}{\partial q}\left(\varkappa V^{-1}\frac{\partial T}{\partial q}\right)+\frac{\partial}{\partial q}\left[u\left(\mu V^{-1}\frac{\partial u}{\partial q}-p\right)\right]$$

Here the unknown functions u, V, p, ε and T are, respectively, the velocity, specific volume, pressure, internal energy and absolute temperature; μ and $\varkappa$ are the viscosity and heat conductivity coefficients, τ is time and q denotes the Lagrangian mass coordinate. The system is closed by two equations of state which for a polytropic perfect gas can be written in the form

$$pV=RT,\quad \varepsilon=c_V T,$$

where $R>0$ is the gas constant and c_V is heat capacity at constant volume. We assume that μ, $\varkappa$ and c_V are positive constants and, that the region occupied by the gas is bounded: $0\leqslant q\leqslant q_1<\infty$. The distribution of u, V and T is assumed known at the initial instant $\tau=0$; the boundaries $q=0$ and $q=q_1$ are, by definition, impermeable and thermally insulated

$$u=u_0(q),\quad V=V_0(q),\quad T=T_0(q)\quad \text{for}\quad \tau=0,\quad 0<q<q_1 \tag{1.2}$$
$$u=0,\quad \partial T/\partial q=0\quad \text{for}\quad q=0,\quad q=q_1$$

In addition, $V_0(q)$ and $T_0(q)$ are positive and bounded functions. Let us introduce the dimensionless variables

$$x=\frac{q}{q_1},\quad t=\frac{\tau}{\tau_1},\quad \rho=\frac{V_1}{V},\quad v=\frac{u}{u_1},\quad \theta=\frac{T}{T_1}$$

$$\left(V_1 = \frac{1}{q_1}\int_0^{q_1} V_0(q)\,dq,\ \tau_1 = \frac{q_1^2 V_1}{\mu},\ u_1 = \frac{\mu}{q_1},\ T_1 = \frac{\mu^2}{q_1^2 c_V}\right)$$

Then the domain of variation of x reduces to the unit segment $\overline{\Omega} = [0, 1]$ and the system of equations assumes the form

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left(\rho \frac{\partial v}{\partial x}\right) - k \frac{\partial}{\partial x}(\rho\theta) \tag{1.3}$$

$$\frac{\partial \theta}{\partial t} = \lambda \frac{\partial}{\partial x}\left(\rho \frac{\partial \theta}{\partial x}\right) + \rho\left(\frac{\partial v}{\partial x}\right)^2 - k\rho\theta \frac{\partial v}{\partial x}$$

$$\left(k = \frac{R}{c_V},\ \lambda = \frac{\varkappa}{\mu c_V}\right)$$

Here the third equation represents the difference between the third equation of (1.1) and the second equation multiplied by u The boundary and initial conditions are written in the form

$$v = \partial\theta/\partial x = 0 \quad \text{for} \quad x = 0, \quad x = 1 \tag{1.4}$$

$$v = v_0(x), \quad \rho = \rho_0(x), \quad \theta = \theta_0(x) \quad \text{for} \quad t = 0, \quad x \in \Omega \tag{1.5}$$

and the functions $\rho_0(x)$ and $\theta_0(x)$ are strictly positive and bounded

$$m = \min\{\inf_\Omega \rho_0(x),\ \inf_\Omega \theta_0(x)\} > 0 \tag{1.6}$$

$$M = \max\{\sup_\Omega \rho_0(x),\ \sup_\Omega \theta_0(x)\} < \infty$$

In addition, in the dimensionless variables the initial density $\rho_0(x)$ has the property

$$\int_0^1 \rho_0^{-1}(x)\,dx = 1 \tag{1.7}$$

Multiplying the second equation of (1.3) by $v(x, t)$ and adding it to the third equation, we obtain

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(\rho \frac{\partial w}{\partial x}\right) + (\lambda - 1)\frac{\partial}{\partial x}\left(\rho \frac{\partial \theta}{\partial x}\right) - k \frac{\partial}{\partial x}(\rho\theta v) \tag{1.8}$$

$$(w(x, t) = \theta(x, t) + {}^1/_2 v^2(x, t))$$

The aim of this paper is to prove the solvability of the system (1.3)—(1.5) in the rectangle $Q = \Omega \times (0, h)$ of arbitrary finite height h, $0 < h < \infty$.

Let us now establish the meaning of the solution of the problem. We denote by $\|\cdot\|$ the norm in $L_2(\Omega)$ and for the remaining spaces the symbol denoting the norm will be accompanied by the relevant index. At times the functions of two variables will be treated as functions of the argument t with the values belonging to a Banach space. The constants depending only on the initial values (1.5) of the parameters λ, h, k, m, M and on the constants of the inclusion theorems, will be denoted by N with appropriate indices.

Definition. We shall call the set of functions (ρ, v, θ)

$$\rho(t) \in L_\infty(0, h; W_2^1(\Omega)), \quad \frac{\partial \rho}{\partial t}(t) \in L_\infty(0, h; L_2(\Omega))$$

$$(v(t), \theta(t)) \in L_\infty(0, h; W_2^1(\Omega)) \cap L_2(0, h; W_2^2(\Omega)) \cap$$

$$\cap\, W_2^1(0, h; L_2(\Omega))$$

a generalized solution of the problem (1.3)—(1.5). The functions satisfy the equations of the system almost everywhere in $Q = \Omega \times (0, h)$, and assume the prescribed initial and boundary values in the sense of the traces of the functions belonging to the above classes.

2. Basic theorems and the scheme of their proof.

Theorem 1. Let the initial values satisfy the conditions (1.6), (1.7) and

$$(\rho_0(x), v_0(x), \theta_0(x)) \in W_2^1(\Omega), \quad v_0(0) = v_0(1) = 0$$

Then a unique generatized solution of the problem (1.3)—(1.5) exists, the functions $\rho(x, t)$ and $\theta(x, t)$ are strictly positive and bounded.

Theorem 2. If in addition to the requirements of Theorem 1 the conditions

$$(v_0, \theta_0) \in C^{2+\alpha}(\Omega), \quad \rho_0 \in C^{1+\alpha}(\Omega), \quad 0 < \alpha < 1$$

also hold and the initial values are consistent with the boundary values

$$v_0 = \theta_0' = 0, \quad (\rho_0 v_0')' - k(\rho_0\theta_0)' = 0 \quad \text{for} \quad x = 0, \quad x = 1$$

then the solution of the problem is classical

$$(v(x, t), \theta(x, t)) \in C^{2+\alpha, 1+\alpha/2}(Q), \quad \rho(x, t) \in C^{1+\alpha, 1+\alpha/2}(Q)$$

Proof of the theorems is based on the use of a priori estimates the constants in which depend only on the data of the problem and on the height h of the rectangle Q. The estimates make it possible to continue the local solution the existence of which can be proved using the principle of compressed mappings, to the whole interval $[0, h]$. Without going into the proof of the local solvability of the problem, we shall show that the operator equation equivalent to the problem is constructed by linearization of Eq. (1.3). Since the operator obtained is contractive on a small time interval, we can apply the Banach theorem.

The first (energetic) estimate can be obtained by integrating Eq. (1.8) in Ω, with the conditions (1.4) taken into account

$$\frac{d}{dt}\int_0^1 w(x, t)\, dx = \frac{d}{dt}\int_0^1 \left[\theta(x, t) + \frac{1}{2} v^2(x, t)\right] dx \equiv 0 \qquad (2.1)$$

This yields

$$\|\theta(t)\|_{L_1(\Omega)} + \frac{1}{2}\|v(t)\|^2 = \|\theta_0\|_{L_1(\Omega)} + \frac{1}{2}\|v_0\|^2 \equiv N_0 < \infty \qquad (2.2)$$

The above equation is valid as long as $\theta(x, t) \geqslant 0$. Therefore in the next stage we verify the positiveness of the temperature obtaining, at the same time, the upper bound for the density. Next we prove the strict positiveness of the density. In deriving these estimates we use a number of auxilliary lemmas which will be formulated in Sect. 3. In the final part we prove the estimates for the derivatives of the functions in question, and study the differential properties of the solutions.

3. Auxiliary assumptions. First we note two simple properties of the density $\rho(x, t)$.

Lemma 1. If $\rho(x, t)$ is a function positive and continuous in $\bar{Q}$, then the following equation holds for every $t \in [0, h]$:

$$\int_0^1 \rho^{-1}(x, t)\, dx = 1 \qquad (3.1)$$

and at least one point $a = a(t) \in [0, 1]$ exists such that

$$\rho(a(t), t) = 1, \quad \forall t \in [0, h] \tag{3.2}$$

Proof. Let us write the first equation of (1.3) in the form $(\rho^{-1})_t = v_x$ and integrate it over Ω. Taking into account the conditions (1.4) and the property (1.7) of the function $\rho_0^{-1}(x)$ we obtain, as the result, the formula (3.1). According to our assumption, the function $\rho(x, t)$ is continuous, therefore the second assertion of the lemma obviously follows from (3.1).

Let us derive yet another corollary of the system (1.3). We eliminate the quantity $\rho \partial v / \partial x = -\partial \ln \rho / \partial t$ from the second equation of (1.3) and integrate the resulting expression with respect to t

$$\frac{\partial}{\partial x}\left[\ln \rho(x, t) + \int_0^t p(x, \tau)\, d\tau\right] = \frac{d \ln \rho_0(x)}{dx} + v_0(x) - v(x, t) \tag{3.3}$$

$$(p(x, t) = k\rho(x, t)\theta(x, t))$$

Performing the second integration at fixed t from the point $a(t)$ where $\rho(a(t), t) = 1$ to an arbitrary $x \in [0, 1]$, and taking antilogs, we obtain

$$\rho(x, t) \exp\left\{\int_0^t p(x, \tau)\, d\tau\right\} = \rho_0(x)\, Y(t)\, B(x, t) \tag{3.4}$$

$$Y(t) = \rho_0^{-1}(a(t)) \exp\left\{\int_0^t p(a(t), \tau)\, d\tau\right\} \tag{3.5}$$

$$B(x, t) = \exp\left\{\int_{a(t)}^{x} [v_0(\xi) - v(\xi, t)]\, d\xi\right\} \tag{3.6}$$

Let us now multiply both sides of the formula (3.4) by $k\theta(x, t)$ and use the definition of the function $p(x, t)$ as given within the brackets in (3.3). Integrating from 0 to any value of t, we obtain

$$\exp\left\{\int_0^t p(x, \tau)\, d\tau\right\} = 1 + k\rho_0(x)\, I(x, t)$$

$$I(x, t) = \int_0^t Y(\tau)\, B(x, \tau)\, \theta(x, \tau)\, d\tau$$

Consequently, the formula (3.4) assumes the form

$$\rho(x, t) = Y(t) B(x, t) \left[\rho_0^{-1}(x) + k\, I(x, t)\right]^{-1} \tag{3.7}$$

It can be shown that the functions $Y(t)$ and $B(x, t)$ appearing in the above expression are strictly positive and bounded.

Lemma 2. The following uniform estimates hold under the conditions of Theorem 1:

$$0 < N_1^{-1} \leqslant B(x, t) \leqslant N_1 < \infty, \quad N_1^{-1} \leqslant Y(t) \leqslant N_2 < \infty \tag{3.8}$$

Proof. Applying the Hölder inequality, we obtain from (2.2)

$$\left| \int_{a(t)}^{x} v(\xi, t)\, d\xi \right| \leqslant \| v(t) \|_{L_1(\Omega)} \leqslant \| v(t) \| \leqslant (2N_0)^{1/2}, \quad \forall (x, t) \in \overline{Q}$$

Therefore the first relation of (3.8) holds with the constant

$$N_1=\exp\{\|v_0\|_{L_1(\Omega)}+(2N_0)^{1/2}\}$$

In proving the estimates for the function $Y(t)$, we can write (3.7) in the following form:

$$Y(t)\rho^{-1}(x,t)=B^{-1}(x,t)\,[\rho_0^{-1}(x)+kI(x,t)]$$

and integrate it over Ω, taking Eq, (3.1) into account. Using the estimates for $B(x,t)$ and the property (1.7) of the initial density $\rho_0(x)$, we arrive at the inequalities

$$N_1^{-1}+kN_1^{-2}\int_0^t Y(\tau)\int_0^1\theta(x,\tau)\,dx\,d\tau\leqslant Y(t)\leqslant N_1+kN_1^2\int_0^t Y(\tau)\int_0^1\theta(x,\tau)\,dx\,d\tau$$

Since

$$\theta(x,t)\geqslant 0,\quad \int_0^1\theta(x,\tau)\,dx\leqslant N_0$$

we conclude that

$$0<N_1^{-1}\leqslant Y(t)\leqslant N_1+kN_0N_1^2\int_0^t Y(\tau)\,d\tau$$

and the proof of the estimates (3.8) is completed by applying the Gronwall lemma.

The formulas (3.7) and (3.8) make it possible to establish important relations connecting the density with the temperature. Let us introduce the following abbreviated notation for the maximum and minimum values of $\rho(x,t)$ and $\theta(x,t)$ at the cross sections $t=\text{const}$:

$$m_\rho(t)=\min_{0\leqslant x\leqslant 1}\rho(x,t),\quad m_\theta(t)=\min_{0\leqslant x\leqslant 1}\theta(x,t) \tag{3.9}$$

$$M_\rho(t)=\max_{0\leqslant x\leqslant 1}\rho(x,t),\quad M_\theta(t)=\max_{0\leqslant x\leqslant 1}\theta(x,t)$$

Lemma 3. For the above qunatities we have the following relations:

$$M_\rho(t)\leqslant N\left[1+n\int_0^t m_\theta(\tau)\,d\tau\right]^{-1} \tag{3.10}$$

$$m_\rho(t)\geqslant n\left[1+N\int_0^t M_\theta(\tau)\,d\tau\right]^{-1} \tag{3.11}$$

$$N=N_1N_2\max\{M,km\},\quad n=N_1^{-2}\min\{\kappa M,m\}$$

Proof of these inequalities follows from the formula (3.7), provided that the estimates (3.8) for $B(x,t)$ and $Y(t)$ are taken into account.

Lemma 4. The following inequality holds for any $\eta>0$:

$$M_\theta^2(t)\leqslant\eta J_1(t)+C_\eta J_2(t)+K_\eta \tag{3.12}$$

$$J_1(t)=\int_0^1\rho(x,t)\left[\frac{\partial\theta}{\partial x}(x,t)\right]^2dx,\quad J_2(t)=\int_0^t J_1(\tau)\,d\tau$$

with constants C_η and K_η depending on the values of h and η of the problem.

Proof. The following relations hold:

$$|\psi(x,t)|^{3/2}=\frac{3}{2}\int_{x_1}^{x}|\psi(\xi,t)|^{1/2}\operatorname{sign}\psi(\xi,t)\frac{\partial\psi(\xi,t)}{\partial\xi}\,d\xi$$

$$\psi(x,t) \equiv \theta(x,t) - \int_0^1 \theta(\xi,t)\,d\xi \quad \left(\int_0^1 \psi(x,t)\,dx = 0\right)$$

$$x_1 = x_1(t) \in [0,1], \quad \psi(x_1(t),t) = 0$$

We estimate the integral in the right-hand side with the help of the Cauchy inequality, taking the first cofactor with the weight $\rho^{-1/2}(\xi,t)$

$$|\psi(x,t)|^{3/2} \leqslant \frac{3}{2}\left(\int_0^1 \rho^{-1}(\xi,t)\,|\psi(\xi,t)|\,d\xi\right)^{1/2}\left(\int_0^1 \rho(\xi,t)\left(\frac{\partial\psi(\xi,t)}{\partial\xi}\right)^2 d\xi\right)^{1/2}$$

Since

$$\psi_\xi = \theta_\xi, \quad \rho^{-1}(\xi,t) \leqslant m_\rho^{-1}(t), \quad \int_0^1 |\psi(\xi,t)|\,d\xi \leqslant 2N_0$$

we find that

$$|\psi(x,t)|^{3/2} \leqslant 3\left(\frac{N_0}{2}\right)^{1/2} m_\rho^{-1/2}(t)\, J_1^{1/2}(t)$$

Let us raise both sides to the power $4/3$ and strengthen the inequality using the formula (3.11) for $m_\rho(t)$. This gives us

$$M_\theta^2(t) \leqslant N_3 + N_4\left(1 + N\int_0^t M_\theta(\tau)\,d\tau\right)^{2/3} J_1^{2/3}(t)$$

Next we apply the Young and Hölder inequalities to the second term in the right-hand side

$$M_\theta^2(t) \leqslant \eta J_1(t) + N_5\eta^{-2}\int_0^t M_\theta^2(\tau)\,d\tau + N_6 \tag{3.13}$$

from which, according to the Gronwall lemma, follows (3.12).

4. Upper and lower bounds for the density and temperature.

Lemma 5. A constant $m_0 > 0$ exists such that

$$m_\theta(t) \geqslant m_0, \quad \forall t \in [0,h] \tag{4.1}$$

Proof. We add and subtract $\frac{1}{4}k^2\rho\theta^2$ to and from the right-hand side of the last equation of (1.3) and divide both parts by $-\theta^2$, to obtain

$$\frac{\partial\omega}{\partial t} = \lambda\frac{\partial}{\partial x}\left(\rho\frac{\partial\omega}{\partial x}\right) - \left[2\lambda\rho\theta\left(\frac{\partial\omega}{\partial x}\right)^2 + \rho\omega^2\left(\frac{\partial v}{\partial x} - \frac{k}{2}\theta\right)^2\right] + \frac{k^2}{4}\rho, \quad \omega \equiv \theta^{-1}$$

Let us multiply this equation by $2r\omega^{2r-1}$, where r is an arbitrary natural number, and integrate over Ω. Taking into accunt the fact that the expression in the square brackets is nonnegative, we obtain the inequality

$$\|\omega(t)\|_{L_{2r}(\Omega)}^{2r-1}\frac{d}{dt}\|\omega(t)\|_{L_{2r}(\Omega)} \leqslant \frac{k^2}{4}\int_0^1 \rho\,\omega^{2r-1}\,dx$$

Let us apply the Hölder inequality to the right-hand side, reduce the result by $\|\omega(t)\|_{L_{2r}(\Omega)}^{2r-1}$ and integrate from 0 to an arbitrary t. At the limit as $r \to \infty$ we obtain, in terms of the notation (3.9), the relation

$$m_\theta^{-1}(t) \leqslant m^{-1} + \frac{k^2}{4}\int_0^t M_\rho(\tau)\,d\tau$$

We reinforce it with the help of the inequality (3.10) for $M_\rho(\tau)$

$$m_\theta^{-1}(t) \leqslant y(t), \quad y(t) = m^{-1} + \frac{Nk^2}{4}\int_0^t \left[1 + n\int_0^\tau m_\theta(s)\,ds\right]^{-1} d\tau$$

Integrating this inequality with respect to $y(t)$ we arrive at the estimate (4.1). The latter implies that from (3.10) follows the boundedness of the density

$$M_\rho(t) \leqslant N, \quad \forall t \in [0, h] \tag{4.2}$$

Lemma 6. A constant $n_0 > 0$ exists such that

$$m_\rho(t) \geqslant n_0, \qquad \forall t \in [0, h] \tag{4.3}$$

We shall begin the proof by verifying the boundedness of the integral $J_2(t)$. Then Lemma 4 will imply the summability on $[0, h]$ of the function $M_\theta(t)$, and the strict positiveness of $m_\rho(t)$ will then follow from the inequality (3.11).

Let us now turn our attention to (1.8). Multiplying it by $w(x, t)$ and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2 + \int_0^1 \rho w_x^2 dx + (\lambda - 1)\int_0^1 \rho\theta_x w_x dx = \tag{4.4}$$

$$k\int_0^1 \rho\theta v w_x dx \leqslant \delta\int_0^1 \rho w_x^2 dx + \frac{k^2}{4\delta}\int_0^1 \rho\theta^2 v^2 dx$$

where $\delta > 0$ is arbitrary. By definition, the functions w $(1 - \delta)w_x^2 + (\lambda - 1)\theta_x w_x \geqslant (\lambda - 2\delta)\theta_x^2 - [{}^1/_4\delta^{-1}(1 + \lambda)^2 + 2\delta - \lambda - 2]v^2 v_x^2$, therefore choosing $\delta = {}^1/_8 \min(1, \lambda)$ we can strengthen the inequality (4.4)

$$\frac{d}{dt}\|w(t)\|^2 + \frac{3\lambda}{2}J_1(t) \leqslant N_6 F(t) + N_7 G(t) \tag{4.5}$$

$$F(t) = \int_0^1 \rho\theta^2 v^2 dx, \quad G(t) = \int_0^1 \rho v^2 v_x^2\, dx$$

Multiplying the second equation of (1.3) by $v^3(x, t)$ and integrating by parts over Ω yields, with the help of the Cauchy inequality,

$$\frac{1}{4}\frac{d}{dt}\|v(t)\|^4_{L_4(\Omega)} + 3G(t) = 3\int_0^1 \rho\theta v^2 v_x dx \leqslant \frac{3}{2}G(t) + \frac{3}{2}F(t) \tag{4.6}$$

Combining the inequalities (4.5) and (4.6), we obtain

$$\frac{d}{dt}(\|w(t)\|^2 + \gamma\|v(t)\|^4_{L_4(\Omega)}) + \frac{3\lambda}{2}J_1(t) \leqslant N_8 F(t)$$

$$(\gamma = {}^1/_6 N_7, \quad N_8 = N_6 + N_7)$$

The right-hand side of the above formula does not exceed $2N_0 N N_8 M_\theta^2(t)$, since $\rho(x, t) \leqslant N$ and $\|v(t)\|^2 \leqslant 2N_0$. The estimate (3.12) in turn holds for M_θ^2, consequently

$$\frac{d}{dt}(\|w(t)\|^2 + \gamma\|v(t)\|^4_{L_4(\Omega)}) + \frac{3\lambda}{2}J_1(t) \leqslant 2N_0 N N_8[\eta J_1(t) + C_\eta J_2(t) + K_\eta] \tag{4.7}$$

We choose $\eta > 0$ so that $4N_0 N N_8 \eta = \lambda$. Then, for the positive function

$$z(t) = \| w(t) \|^2 + \gamma \| v(t) \|^4_{L_4(\Omega)} + \lambda J_2(t)$$

the relation (4.7) yields the following differential inequality:

$$\frac{dz}{dt} \leqslant N_9 z + N_{10}$$

From this we conclude that $z(t)$ is a function bounded on $[0, h]$, i.e.

$$J_2(t) \leqslant N_{11}, \quad \forall\, t \in [0, h] \tag{4.8}$$

$$\| w(t) \|^2 + \gamma \| v(t) \|^4_{L_4(\Omega)} \leqslant N_{12} \tag{4.9}$$

By virtue of the inequality (3.12), (4.8) implies the summability of the function $M_\theta^2(t)$ in $[0, h]$, and even more strongly, that of $M_\theta(t)$

$$\int_0^h M_\theta(t)\, dt \leqslant h^{1/2} \left(\int_0^h M_\theta^2(t)\, dt \right)^{1/2} \leqslant N_{13} \tag{4.10}$$

After this, the formula (3.11) yields the following lower bound for $m_\rho(t)$

$$m_\rho(t) \geqslant n\,[1 + NN_{13}]^{-1} \equiv n_0 > 0, \quad \forall\, t \in [0, h]$$

which completes the proof of Lemma 6.

We note two of its properties. Firstly, in accordance with the definition of $w(x, t)$ we have from (4.9)

$$\max_{0 \leqslant t \leqslant h} \| \theta(t) \| \leqslant N_{14} < \infty \tag{4.11}$$

Secondly, the inequality (4.3) together with (4.8) yields the estimate

$$\int_0^h \| \theta_x(t) \|^2 dt \leqslant N_{15} \tag{4.12}$$

5. Estimate for the derivatives of the unknown functions. Using the inequalities obtained above, we shall prove the remaining a priori estimates indicated in Theorems 1 and 2. At this stage we shall follow a scheme given in [3, 6] and show only the corresponding necessary changes in the method illustrating the derivation of the estimates with examples.

Multiplying the second equation of (1.3) by $v(x, t)$, and integrating over Ω, we find

$$\int_0^h \| v_x(t) \|^2 dt \leqslant N_{16} \tag{5.1}$$

Further, differentiating (3.7) with respect to x, we obtain

$$\frac{\partial \rho(x, t)}{\partial x} = \rho(x, t) A(x, t) + \rho^2(x, t) B^{-1}(x, t) Y^{-1}(t) \left\{ \frac{d\rho_0^{-1}(x)}{dx} - k \int_0^t B(x, \tau) Y(\tau) \left[\frac{\partial \theta(x, \tau)}{\partial x} + \theta(x, \tau) A(x, \tau) \right] d\tau \right\} \tag{5.2}$$

$$(A(x, t) = v_0(x) - v(x, t))$$

From this we obtain, using the estimates (2.2), (3.8) and (4.10)—(4.12),

$$\max_{0 \leqslant t \leqslant h} \left\| \frac{\partial \rho}{\partial x}(t) \right\| \leqslant N_{17} \tag{5.3}$$

Next we multiply the second equation of (1.3) by v_{xx} and integrate over Ω. The inequalities (4.2), (4.3), (4.11) and (5.3) together yield the estimate

$$\max_{0\leqslant t\leqslant h}\|v_x(t)\|^2+\int_0^h\|v_{xx}(t)\|^2\,dt\leqslant N_{18} \tag{5.4}$$

Now from (1.3) we obtain directly

$$\max_{0\leqslant t\leqslant h}\left\|\frac{\partial\rho}{\partial t}(t)\right\|\leqslant N_{19},\quad \int_0^h\|v_t(t)\|^2\,dt\leqslant N_{20} \tag{5.5}$$

Similarly, multiplying the third equation of (1.3) by θ_{xx}, we deduce

$$\max_{0\leqslant t\leqslant h}\|\theta_x(t)\|^2+\int_0^h\|\theta_{xx}(t)\|^2\,dt+\int_0^h\|\theta_t(t)\|^2\,dt\leqslant N_{21} \tag{5.6}$$

Differentiating the first equation of (1.3) with respect to x and taking (5.4) into account, we also obtain

$$\int_0^h\|\rho_{xt}(t)\|^2\,dt\leqslant N_{22} \tag{5.7}$$

This concludes the proof of the estimates appearing in Theorem 1. We note that the inequalities (5.3), (5.5) and (5.7) guarantee, by virtue of the embedding theorem the Hölder continuity of the density $\rho(x,t)$ with the index $1/2$.

Passing now to the estimates of Theorem 2, we shall first show that $v(x,t)$ and $\theta(x,t)$ are also Hölder continuous. To do this we differentiate the second and third equation of (1.3) with respect to t, and we multiply the results by v_t and θ_t, respectively. Using the previously obtained inequalities, we arrive at the estimates

$$\max_{0\leqslant t\leqslant h}\|v_t(t)\|^2+\int_0^h\|v_{xt}(t)\|^2\,dt\leqslant N_{23} \tag{5.8}$$

$$\max_{0\leqslant t\leqslant h}\|\theta_t(t)\|^2+\int_0^h\|\theta_{xt}(t)\|^2\,dt\leqslant N_{24}$$

The above estimates together with (5.4) and (5.6) imply, by virtue of the embedding theorem, the Hölder continuity in Q of the functions $v(x,t)$ and $\theta(x,t)$ with the index $1/2$. Further, from the formula (5.2) follows the Hölder continuity in Q of the derivative $\partial\rho/\partial x$. In fact, the first term in the right-hand side of (5.2) is Hölder continuous by virtue of the properties of ρ, v_0 and v. The cofactor $\rho(x,t)B^{-1}(x,t)Y^{-1}(t)$ has this property in accordance with (3.7). Finally, the integral

$$\int_0^t\frac{\partial\theta}{\partial x}(x,\tau)\,d\tau$$

is also Hölder continuous in $\overline{Q}$ by virtue of the estimates (5.6).

Having proved the Hölder continuity of the density $\rho(x,t)$ and of its derivative with respect to x, we can consider the second and third equations of (1.3) as a system parabolic in $v(x,t)$ and $\theta(x,t)$, with Hölder coefficients and with right-hand side. Using the estimates of solutions of the parabolic equations in the classes of Hölder functions, we raise the smoothness of the solution to that indicated in Theorem 2.

Note. In the same manner we can show the solvability of a problem with the boundary conditions of the form

$$v = \theta = 0 \quad \text{for} \quad x = 0, x = 1$$

We assume here that the initial temperature $\theta_0(x)$ is nonnegative. Then considering the third equation of (1.3) as linearly parabolic in $\theta(x, t)$, we conclude with the help of the maximum principle that $\theta(x, t) \geqslant 0$ in $\bar{Q}$. This property of the temperature enables us to use the inequalities (3.10) and (4.2) in the process of obtaining the upper bound for the density.

We must introduce yet another change into the derivation of the first (energetic) estimate. Let us multiply the second equation of (1.3) by v and the third equation by $\theta(\theta^2 + \delta)^{-1/2}$, where $\delta > 0$ is an arbitrary number. Adding and integrating, we obtain

$$\frac{d}{dt}\int_0^1 \left[(\theta^2 + \delta)^{1/2} + \frac{1}{2} v^2\right] dx \leqslant \frac{k^2}{4}\int_0^1 \rho\theta^2 [1 - \theta(\theta^2 + \delta)^{-1/2}]\, dx$$

Integrating now with respect to t and making δ tend to zero, we arrive at the required estimate

$$\|\theta(t)\|_{L_1(\Omega)} + \frac{1}{2}\|v(t)\|^2 \leqslant N_0 = \|\theta_0\|_{L_1(\Omega)} + \frac{1}{2}\|v_0\|^2$$

The remaining arguments appearing in the problem with specified values of the temperature at the boundaries are the same as those considered in the problem with a given heat flux which has been studied in detail.

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